Chapter 3

Fundamental Maths

To be clear that this section is to introduce the meaning of Maths linked to real problems to be solved in 3D vision, but not intended to give all details of Maths. For a coherent understanding of Maths, please refer to Calculus 1 and 2 of Gilbert Strang [17].
Whole story is to solve the basic linear equation

$$AX = B \tag{3.1}$$

Eq.3.1 can be simple or complex depending on its dimensionality and non-linearity, even though it is linearised in a linear representation form. Further more, in many cases, it is not possible to have an absolute solution, but a rough solution of minimization

$$Min_X ||AX - B||^2 \tag{3.2}$$

Step by step we will explain why many real problems can be formulated in a linear form of linear algebra or being approximated through a linearising process. All ends up with the same optimization Eq.3.2.

Real-life example
On Christmas day (24.12.2018), we together went by train to visit Frankfurt Shopping Centre, Germany. Onward we paid 7 Euro for train tickets in which 2 euro per child and 3 Euro per adult. In the way back, after 12:00, there is a discount from RMW-train company that each ticket is 1 Euro for everyone as a Christmas gift. Whereby, we only needed to pay 3 Euro for all. Can you help to figure out how man children? How many adults?

Solution:

Let denote x as a number of children, and y as a number of adults. Simple linear equations can be derived

$$2x + 3y = 7x + y = 3$$

It is trivial to have roots as follow

$$x = 2y = 1$$

In matrix representation, we can establish

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$B = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Then,

$$AX = B \equiv \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

We have $X = A^{-1}AX = A^{-1}B$. How to find the inverse A^{-1} will be discussed later, let assume that a predetermined calculation gives

$$A^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Matrix representation helps to have a nice expression when having a higher dimensionality. For example, n linear equations to be solved for n variables $X = \{x_1, x_2, .., x_n\}$, $B = \{b_1, b_2, .., b_n\}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}$$

We still come back to the familiar form: AX = B.

3.1 Remarks on Matrix

An example of 2x2 Matrix A as follow

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant of A

$$det(A) = |ad - bc|$$

Transpose of A

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Inverse of A

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Jacobian matrix

$$J_f = [\frac{\delta f}{\delta x_1} \quad \cdots \quad \frac{\delta f}{\delta x_n}] = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \cdots & \frac{\delta f_1}{\delta x_n} \\ \cdots & \cdots & \cdots \\ \frac{\delta f_n}{\delta x_1} & \cdots & \frac{\delta f_n}{\delta x_n} \end{bmatrix}$$

Matrix Differentiation

$$\frac{\delta(u^T v)}{\delta x} = u^T \frac{\delta v}{\delta x} + v^T \frac{\delta u}{\delta x}$$

It is trivial to obtain

$$\frac{\delta f}{\delta x} = x^T (A + A^T)$$

where

$$f = x^T A x$$

Dot Product of two vectors: $u(u_1, u_2, u_3)$ and $v(v_1, v_2, v_3$

$$u.v = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{3.3}$$

Cross product

$$u \times v = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \tag{3.4}$$

Or: $u \times v = (u_2 v_3 - u_3 v_2)\vec{i} + (u_3 v_1 - u_1 v_3)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$ where $(\vec{i}, \vec{j}, \vec{k})$ are basic unit vectors of the working coordinate.

For a convenient in expression, we also denote

$$u_\times = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{3.5}$$

So that $u \times v = u_\times v$

3.1.1 Inverse of a matrix A

A very common way of finding A^{-1} is to use Row Operation, so called **Gauss-Jordan**. The basic idea is that if a transformation P which transforms $[A|I]$ to $[U|V]$

$$P * [A|I] = [U|V]$$

Or

$$P * A = U$$
$$P * I = V$$

If $U = I$, then

$$P = A^{-1}$$
$$P = V$$

Therefore, we need to use Row Operation to transform $[A|I]$ to be $[I|V]$, and thus $A^{-1} = V$. For example,

$$[A|I] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

row1 = -1*(row1 - 3* row2)

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

row2 = row2 - row1

$$[A|I] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

Therefore

$$A^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

Note that not all matrix has an inverse, for example

$$A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$det(A) = 0$, so called a Singular Matrix.

3.1.2 Null Space

Definition 1. The null-space of $m \times n$ matrix A, denoted *Null A*, is the set of all solutions to the homogeneous equation $Av = 0$.

Intuitively, if mapping V $\rightarrow$ W. Null space or Kernel L is mapping to O.

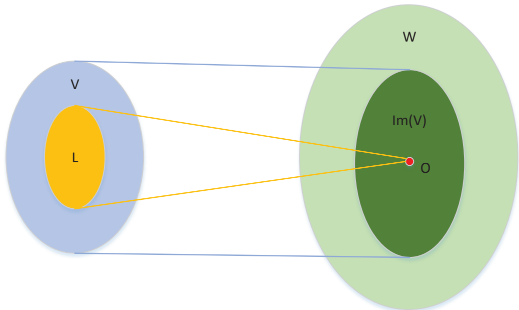


Figure 3.1: Mapping V to W results in Im(V) where the image of L is just one point O.

$$kerL = \{v \in V | l(v) = 0\} \tag{3.6}$$

$dim(kerL) + dim(imL) = dim(V)$

Assume that (v_1, v_2) are roots of Eq.3.1, then

$$A * v_1 = BA * v_2 = B$$

hence, $A(v_1 - v_2) = 0$ is the root of null-space. Therefore, if we can find $v = \{v_1, v_2, .., v_n\}$ are the roots of null-space, and a specific root X_0 of Eq.3.1. The roots of Eq.3.1 can be found as $\{X_i\} = X_0 + \{v_i\}$.

Finding a Null space

Come back to a similar idea in finding the inverse of a matrix, let consider a transformation P which transforms $[A|I]$ to $[U|V]$

$$P * [A|I] = [U|V]$$

Or

$$P * A = U$$
$$P * I = V$$

Thus $P = V$, then $v_i^T A^T = u_i^T$ or $Av_i = u_i$. Therefore, if $u_i = 0$ then $Av_i = 0 \rightarrow v_i$ is a basic of Null space.

3.1.3 Orthogonal and Orthonormal

Orthogonal: $U^T V = 0$
Orthonormal: $U^T V = 0$ and $U^T U = V^T V = I$

3.2 Eigenvalue and Eigenvector

Definition 2. Eigenvector: Special vector which multiplies with A will result in its scaling, but no direction change.

$$Av = \lambda v \tag{3.7}$$

λ is called eigenvalue. Note that, it is trivial to prove that

$$A^{2n} v = \lambda^{2n} v$$

The meaning of finding eigenvector/eigenvalues is to find a basic formation of matrix A, which helps to be easier or convenient in processing optimization problems related to a matrix operation.

Even though eigenvalue and eigenvector are mentioned in many books and used in many maths-related solutions, a common concern from students "I am not sure that I really understand Eigenvalue and Eigenvector in my real life, how it looks like". To give readers a clear example, let consider a matrix A which transform a Circle into an Ellipse (major axis a, minor axis b, and angle θ). For an easy mathematics representation, **let consider the centre of the Circle is the origin**, otherwise just simply adding the translation of the centre against the origin. Each (x', y') on the Ellipse can be determined through (x, y) on the Circle, expressed as follow,

$$x' = \frac{xcos\theta - ysin\theta}{a}$$

$$y' = \frac{xsin\theta + ycos\theta}{b}$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} acos\theta & -asin\theta \\ bsin\theta & bcos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For each point (x'_i, y'_i) of the Ellipse, corresponding to a point (x_i, y_i) of the Circle, a vector which connects the centre of Ellipse to that point is called v_i. Due to the fact that we choose the origin to coincide to the centre, the vector coordinate will be the same as the coordinate of the point.

Let denote,

$$A = \begin{bmatrix} acos\theta & -asin\theta \\ bsin\theta & bcos\theta \end{bmatrix}$$

$$v_i = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

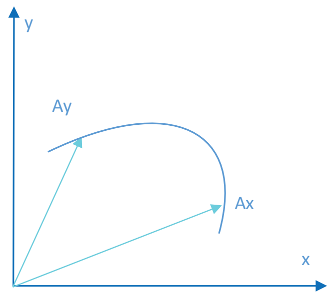
$$X_i = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$v_i = AX_i$$

For each vector X_i, an angle between X_i and the axis x is denoted by θ_i, we have $x_i = cos\theta_i$ and $y_i = sin\theta_i$

$$v_i = AX_i = \begin{bmatrix} acos\theta & -asin\theta \\ bsin\theta & bcos\theta \end{bmatrix} \begin{bmatrix} cos\theta_i \\ sin\theta_i \end{bmatrix} = \begin{bmatrix} acos\theta cos\theta_i - asin\theta \sin\theta_i \\ bsin\theta cos\theta_i + bcos\theta sin\theta_i \end{bmatrix}$$

Thereby, any vector X_i pointing to the same direction of vector u or v, at $\theta_i = -\theta + PI$ or $\theta_i = -\theta$ respectively, multiply with A will result in a scale change, but no direction change. Therefore, (u, v) are eigenvectors of A.



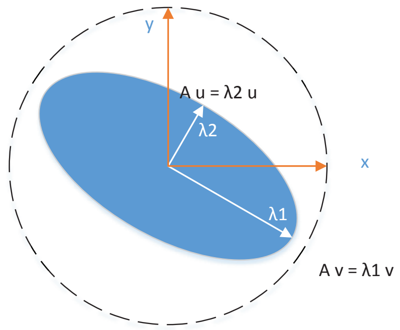


Figure 3.2: a)Left: These are not eigenvectors b)Right: Av lines up with v at eigenvectors

Fig.3.2 shows two examples: Left) Ax and Ay changes the direction of vector x and y respectively. Hence, according to the Definition 2 those are not eigenvector of A; Right) Av and Au line up with v and u respectively, which means multiplying A with u or v only changes its scale, but not the direction of the vector, and thus (u, v) are eigenvectors of A.

A gentle remind of how to find eigenvectors, eigenvalues through the following example,

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$det \begin{bmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{bmatrix} = \lambda^2 - 1.5\lambda + 0.5 = (\lambda-1)(\lambda-0.5) = 0$$

Thus, eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 0.5$. For $\lambda_1 = 1$,

$$0.8u + 0.3v = u$$

$$0.2u + 0.7v = v$$

$$v_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

Similarly

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3.3 Singular Value Decomposition (SVD)

Definition 3. SVD is a factorization of a matrix: $A = U \sum V^T$. Where U, V are orthogonal, and $\sum$ is diagonal.

Why and when do we need an SVD of a matrix? Let come back to our very first aim of solving Eq.3.1: $AX = B$. We need to find the inverse A^{-1}, meanwhile it was mentioned before that not all matrix A exists its inverse matrix. For an ill-conditioned matrix A, a standard approach of finding A^{-1} ends up with different results per process or uncontrollable solution.

What is an ill-conditioned matrix?

Given an invertible matrix A, and thus we have A^{-1} existing. Let come back to our Eq.3.1 with an assumption of having some noise from our collected data, denoted as δb.

$$x = A^{-1}(b + \delta b) = A^{-1}b + A^{-1}\delta b$$

Since $\bar{x} = A^{-1}b$ is the true solution, we have the error of the solution is $\delta x = A^{-1}\delta b$. The error in δb may get amplified by A^{-1} and produce a large error in x. In those situations, where large error is a subjective criterion, we say the problem is ill-posed or ill-conditioned.

In such cases where A^{-1} does not exist or ill-conditioned A, it is better to approximate A^{-1}. This ends up with the idea of SVD. In practice, since SVD is relatively fast in implementation, users might prefer to use SVD regardless of considering if a matrix is ill-conditioned or non-invertible.

First, it is not hard to find an orthogonal basic for the row space, using the Gram-Schmidt process (https://en.wikipedia.org/wiki/Gram. Thereby, we can think of a matrix A as a linear transformation which takes a vector v_i in its row space to a vector u_i in its column space, where $Av_i = \delta_i u_i$.

$$A[v_1 \quad v_2 \quad ... \quad v_n] = [\delta_1 u_1 \quad \delta_2 u_2 \quad ... \quad \delta_n u_n] = [u_1 \quad u_2 \quad ... \quad u_n] \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & ... & \\ & & & \delta_n \end{bmatrix}$$

Or

$$AV = U \sum$$

with $u_1, u_2, ..u_n$ an orthogonal basic of the column space of A, and $v_1, v_2, ..v_n$ an orthogonal basic of the row space of A. Due to orthonormal property $V^T V = I, \quad U^T U = I$, we can derive $A = U \sum V^T$.

We have just explained why a matrix A can be expressed in such form. We now learn why such form is making our life much easier in finding an approximation of A^{-1}.

$$A = U \sum V^T$$

Hence

$$A^T A = V \sum U^T U \sum V^T = V \sum V^T$$

This is in the form of $Q \sum Q^T$, which can be diagonalized to find V. The column of V are eigenvectors of $A^T A$, and the eigenvalues of $A^T A$ are the values δ_i^2. To find U, we do the same thing with AA^T. Please see some numerical examples from Prof. Strang's lectures [17].

Real-life example

Given a set of N 3D points, $\{(x_i, y_i, z_i)\}_{i=1}^N$, your task is to find a best fit plane established from those points. A mathematics expression of a plane is

$$ax + by + cz + d = 0$$

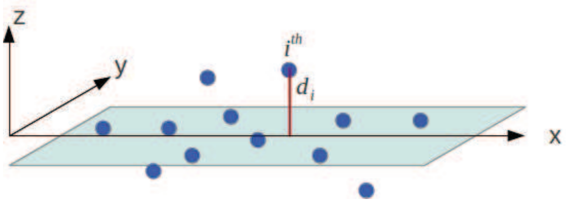


Figure 3.3: Plane Fitting from a given set of 3D points

This is over-determined because the solution space is three-dimensional, but the above description uses four variables. By constraining the solution space, we can assign $c = 1$, hence

$$ax + by + d = -z$$

for all the above N 3D points,

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \dots & \dots & \dots \\ x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -z_1 \\ -z_2 \\ \dots \\ -z_N \end{bmatrix}$$

Let denote

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \dots & \dots & \dots \\ x_N & y_N & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$B = \begin{bmatrix} -z_1 \\ -z_2 \\ \dots \\ -z_N \end{bmatrix}$$

We come back to the familiar linear equation: $AX = B$, and our task is to find X. Applying SVD to have $A = U \sum V^T$, and thus $A^{-1} = V \sum^{-1} U^T$.

$$X = V \sum^{-1} U^T B$$

Applying the above technique to refine the depth information given by a RGBD camera, shown in Fig.3.4

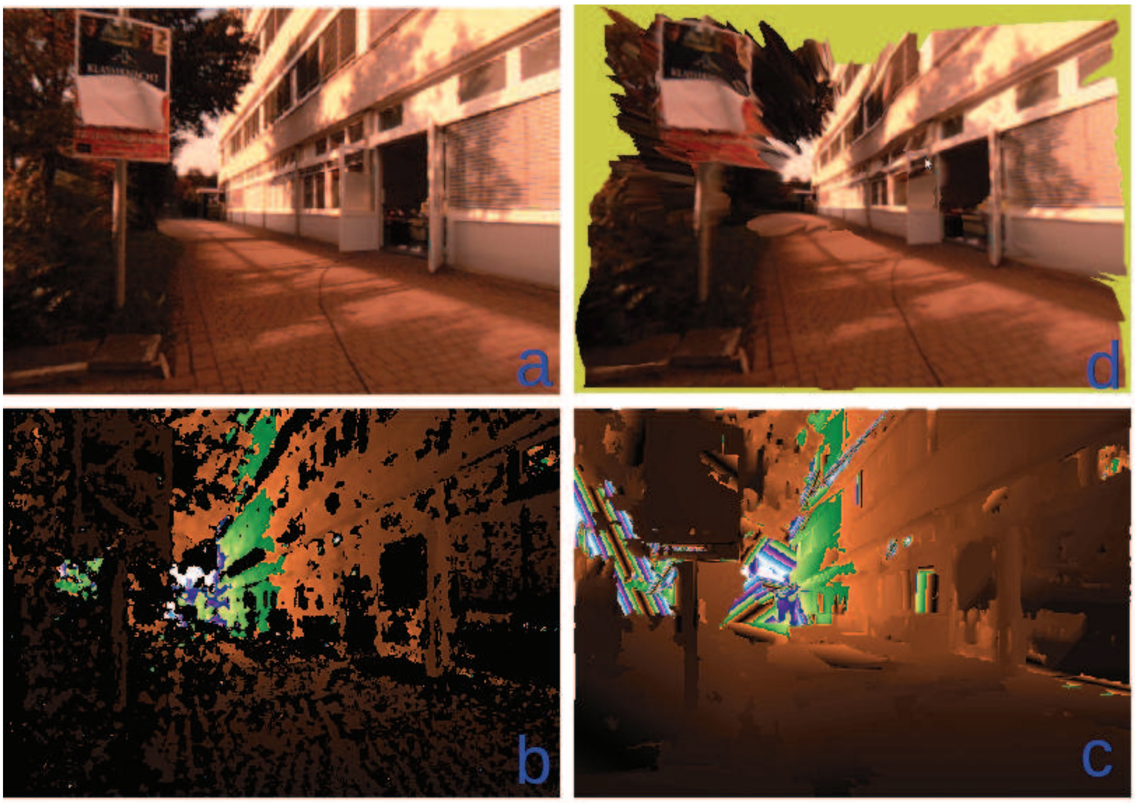


Figure 3.4: Example of Planar Fitting: (b) Given a discrete and noisy set of 3D points by RGBD camera; (c) Least Square Solution using SVD refines and fulfill multiple planar surface to result in a much dense 3D reconstruction of environment; (a) Image of environment; (d) 3D rendering of environment using (a) and (c).